Three-Dimensional Free Vibration Analysis of a Homogeneous Transradially Isotropic Thermoelastic Sphere

J. N. Sharma
e-mail: jns@nitham.ac.in

N. Sharma
e-mail: nivi2j@gmail.com

Department of Applied Sciences,
National Institute of Technology,
Hamirpur 177005, India

In the present paper, an exact three-dimensional vibration analysis of a transradially isotropic, thermoelastic solid sphere subjected to stress-free, thermally insulated, or isothermal boundary conditions has been carried out. Nondimensional basic governing equations of motion and heat conduction for the considered thermoelastic sphere are uncoupled and simplified by using Helmholtz decomposition theorem. By using a spherical wave solution, a system of governing partial differential equations is further reduced to a coupled system of three ordinary differential equations in radial coordinate in addition to uncoupled equation for toroidal motion. Matrix Fröbenious method of extended power series is used to investigate motion along radial coordinate from the coupled system of equations. Secular equations for the existence of various types of possible modes of vibrations in the sphere are derived in the compact form by employing boundary conditions. Special cases of spheroidal and toroidal modes of vibrations of a solid sphere have also been deduced and discussed. It is observed that the toroidal motion remains independent of thermal variations as expected and spheroidal modes are in general affected by thermal variations. Finally, the numerical solution of the secular equation for spheroidal motion (S-modes) is carried out to compute lowest frequency and dissipation factor of different modes with MATLAB programming for zinc and cobalt materials. Computer simulated results have been presented graphically. The analyses may find applications in aerospace, navigation, and other industries where spherical structures are in frequent use. [DOI: 10.1115/1.3172141]

Keywords: toroidal, spheroidal, transradially isotropic, thermoelastic vibrations, spherical structures

1 Introduction

Spherically isotropic materials are of particular importance and use because of their applications in aerospace and nuclear technology [1]. These materials have their physical properties transversely isotropic about the radius vector drawn from some particular point, fixed with respect to the material and hence the elastic properties of such materials are described by five elastic constants. The latest geophysical results reveal that the earth, in fact, should be modeled as a spherically isotropic inhomogeneous sphere with liquid nucleus [2]. Chen [3] reviewed research history of spherically isotropic bodies. According to Schafbuch et al. [4], great achievements were made to obtain general solution of the vibration problems by the authors of Refs. [5–8] for an isotropic sphere. Ding et al. [9] obtained the solutions of vibrational problems of spherical and cylindrical shells with matrix frobenius method. Lamb [10] showed that two basic types of free vibrations, namely, (i) the vibrations with zero volume change and zero radial displacement and (ii) the vibrations with zero radial components of the curl of the displacement exist for an isotropic sphere. These vibrations are referred to as the "vibrations of the first and second classes," respectively. Lapwood and Usami [11] named the first class of vibrations as "torsional" or "toroidal" and the second class as "spheroidal" or "poloidal."

As per knowledge of the authors, no systematic and exact study on the effect of temperature variations on three-dimensional vibration of heat conducting elastic spherical structures is available in literature. Therefore, the purpose of this paper is to present the exact three-dimensional vibration analysis of spherically isotropic, thermoelastic sphere subjected to stress-free, thermally insulated/isothermal boundary conditions. The derived secular equations for spheroidal (S) modes of vibrations, which are dependent on thermal variations, are solved numerically for zinc and cobalt materials in order to compute lowest frequency and dissipation factor. This analysis may be a benchmark to check numerical methods for thermoelastic continuum model analysis.

2 Formulation of the Problem

We consider a homogeneous transversely isotropic thermally conducting, elastic sphere of radius R at uniform temperature T_0 in the undisturbed state initially. The basic governing equations of motion and heat conduction for a linear coupled thermoelastic solid in spherical polar coordinates (r,θ,ϕ,t), in the absence of body forces and heat sources, are given by

$$\sigma_{rr,r} + \frac{1}{r\sin\theta}\sigma_{r\phi,\phi} + \frac{1}{r}\sigma_{r\theta,\theta} + \frac{1}{r}[2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta] = \rho\ddot{u}_r \tag{1}$$

$$\sigma_{r\phi,r} + \frac{1}{r\sin\theta}\sigma_{\phi\phi,\phi} + \frac{1}{r}\sigma_{\phi\theta,\theta} + \frac{1}{r}[3\sigma_{r\phi} + 2\sigma_{\phi\theta}\cot\theta] = \rho\ddot{u}_\phi \tag{2}$$

Contributed by the Applied Mechanics Division of ASME for publication in the Journal of Applied Mechanics. Manuscript received August 30, 2008; final manuscript received February 19, 2009; published online December 9, 2009. Review conducted by Martin Ostoja-Starzewski.

$$\sigma_{r\theta,r}+\frac{1}{r\sin\theta}\sigma_{\phi\theta,\phi}+\frac{1}{r}\sigma_{\theta\theta,\theta}+\frac{1}{r}[3\sigma_{r\theta}+(\sigma_{\theta\theta}-\sigma_{\phi\phi})\cot\theta]=\rho\ddot{u}_\theta \tag{3}$$

$$K_3\left(T_{,rr}+\frac{2}{r}T_{,r}\right)+K_1\left[\frac{1}{r^2}T_{,\theta\theta}+\frac{\cot\theta}{r^2}T_{,\theta}+\frac{1}{r^2\sin^2\theta}T_{,\phi\phi}\right]-\rho C_e\dot{T}$$
$$=T_0[\beta_1(\dot{e}_{\theta\theta}+\dot{e}_{\phi\phi})+\beta_3\dot{e}_{rr}] \tag{4}$$

where

$$\sigma_{\theta\theta}=c_{11}e_{\theta\theta}+c_{12}e_{\phi\phi}+c_{13}e_{rr}-\beta_1 T,\quad \sigma_{r\theta}=2c_{44}e_{r\theta}$$
$$\sigma_{\phi\phi}=c_{12}e_{\theta\theta}+c_{11}e_{\phi\phi}+c_{13}e_{rr}-\beta_1 T,\quad \sigma_{r\phi}=2c_{44}e_{r\phi} \tag{5}$$
$$\sigma_{rr}=c_{13}e_{\theta\theta}+c_{13}e_{\phi\phi}+c_{33}e_{rr}-\beta_3 T,\quad \sigma_{\theta\phi}=2c_{66}e_{\theta\phi}$$

$$e_{rr}=\frac{\partial u_r}{\partial r},\quad e_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r},\quad e_{\phi\phi}=\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}+u_\theta\frac{\cot\theta}{r} \tag{6}$$
$$e_{r\phi}=\frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}+\frac{\partial u_\phi}{\partial r}\right],\quad e_{r\theta}=\frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right]$$
$$e_{\phi\theta}=\frac{1}{2}\left[\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}-\frac{u_\phi\cot\theta}{r}+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right]$$

$$\beta_1=(c_{11}+c_{12})\alpha_1+c_{13}\alpha_3,\quad \beta_3=2c_{13}\alpha_1+c_{33}\alpha_3$$
$$c_{66}=\tfrac{1}{2}(c_{11}-c_{12}) \tag{7}$$

where σ_{ij} and e_{ij} $(i,j=r,\theta,\phi)$ are the stress and strain components, respectively.

Here $\vec{u}=(u_r,u_\theta,u_\phi)$ is the displacement vector; $T(r,\theta,\phi,t)$ is the temperature change; c_{11}, c_{12}, c_{13}, c_{33}, and c_{44} are the five independent isothermal elastic parameters; α_1,α_3 and K_1,K_3 are, respectively, the coefficients of linear thermal expansion and thermal conductivities along and perpendicular to the axis of symmetry; and ρ and C_e are the mass density and specific heat at constant strain, respectively. The comma notation is used for spatial derivatives and the superposed dot denotes time differentiation. It can be proved thermodynamically [12] that $K_1>0$, $K_3>0$ and, of course, $\rho>0$ and $T_0>0$. We assume in addition that $C_e>0$ and the isothermal elasticities are components of a positive definite fourth-order tensor. The necessary and sufficient conditions for the satisfaction of the latter requirements are

$$c_{11}>0,\quad c_{11}>c_{12},\quad {c_{11}}^2>{c_{12}}^2$$
$$c_{44}>0,\quad c_{33}(c_{11}+c_{12})>{c_{13}}^2 \tag{8}$$

We introduce the potential functions ψ, G, and w defined by Ref. [13]:

$$u_\theta=-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}-\frac{\partial G}{\partial\theta},\quad u_\phi=\frac{\partial\psi}{\partial\theta}-\frac{1}{\sin\theta}\frac{\partial G}{\partial\phi},\quad u_r=w \tag{9}$$

Upon using Eq. (9) in Eqs. (1)–(4), we find that ψ, G, w, and T satisfy the equations

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{2}{r^2}+\frac{c_1-c_2}{r^2}+\frac{c_1-c_2}{2r^2}\nabla^2-\frac{\partial^2}{\partial t^2}\right)\psi=0 \tag{10}$$

$$\left(\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\bar{K}}{r^2}\nabla^2-\frac{\partial}{\partial t}\right)T-\varepsilon\frac{\partial}{\partial t}\left[\left(\frac{\partial}{\partial r}+\frac{2\bar{\beta}}{r}\right)w-\frac{\bar{\beta}}{r}\nabla^2 G\right]\right)=0 \tag{11}$$

$$-\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{c_1}{r^2}\nabla^2-\frac{2-c_1+c_2}{r^2}-\frac{\partial^2}{\partial t^2}\right)G$$
$$+\left(\frac{1+c_3}{r}\frac{\partial}{\partial r}+\frac{2+c_1+c_2}{r^2}\right)w-\frac{\bar{\beta}T}{r}=0 \tag{12}$$

$$\left[c_4\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)-\frac{2(c_1-c_3+c_2)}{r^2}+\frac{1}{r^2}\nabla^2-\frac{\partial^2}{\partial t^2}\right]w$$
$$-\left[\frac{1+c_3}{r}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)+\frac{1}{r^2}(2c_3-c_1-c_2)\right]\nabla^2 G$$
$$-\left(\frac{\partial}{\partial r}+\frac{2}{r}-\frac{2\bar{\beta}}{r}\right)T=0 \tag{13}$$

where $\nabla^2=(\partial^2/\partial\theta^2)+\cot\theta(\partial/\partial\theta)+(1/\sin^2\theta)(\partial^2/\partial\phi^2)$. Here we have used the dimensionless quantities

$$r'=\frac{\omega^\bullet}{v_s}r,\quad t'=\omega^\bullet t,\quad u_i'=\frac{\rho\omega^\bullet v_s}{\beta_3 T_0}u,\quad T'=\frac{T}{T_0},\quad \sigma_{ij}'=\frac{\sigma_{ij}}{\beta_3 T_0}$$
$$c_1=\frac{c_{11}}{c_{44}},\quad c_2=\frac{c_{12}}{c_{44}},\quad c_3=\frac{c_{13}}{c_{44}},\quad c_4=\frac{c_{33}}{c_{44}},\quad \bar{K}=\frac{K_1}{K_3} \tag{14}$$
$$\bar{\beta}=\frac{\beta_1}{\beta_3},\quad \varepsilon=\frac{T_0{\beta_3}^2}{\rho C_e c_{44}},\quad \omega'=\frac{\omega}{\omega^\bullet},\quad R'=\frac{\omega^\bullet R}{v_s}$$

where ${v_s}^2=c_{44}/\rho$ and $\omega^\bullet=C_e c_{44}/K_3$ are the shear wave velocity and characteristic frequency of the sphere, respectively. The primes have been suppressed for convenience.

3 Boundary Conditions

We consider the free vibrations of the sphere, which is subjected to two types of boundary conditions at $r=R$ as outlined in Secs. 3.1 and 3.2.

3.1 Mechanical Conditions. The surface $r=R$ of the sphere is assumed to be stress free, which leads to

$$\sigma_{rr}=0,\quad \sigma_{r\theta}=0,\quad \sigma_{r\phi}=0 \tag{15a}$$

3.2 Thermal Conditions. The surface $r=R$ of the sphere is subjected to the following thermal conditions

$$T_r+hT=0 \tag{15b}$$

where h is the surface (Biot's) heat transfer coefficient.

Here $h\to 0$ refers to thermally insulated boundary and $h\to\infty$ corresponds to isothermal surface of the sphere.

4 Solution of the Problem

We can write three displacement functions and the temperature change as

$$\psi(r,\theta,\phi,t)=r^{-1/2}\sum_{n=1}^{\infty}u_n(r)S_n^{\ m}(\theta,\phi)\exp(i\omega t)$$
$$w(r,\theta,\phi,t)=r^{-1/2}\sum_{n=0}^{\infty}w_n(r)S_n^{\ m}(\theta,\phi)\exp(i\omega t) \tag{16}$$
$$G(r,\theta,\phi,t)=r^{-1/2}\sum_{n=1}^{\infty}v_n(r)S_n^{\ m}(\theta,\phi)\exp(i\omega t)$$

$$T(r,\theta,\phi,t) = r^{-1/2}\sum_{n=0}^{\infty} T_n(r) S_n^{\,m}(\theta,\phi)\exp(i\omega t)$$

where $S_n^{\,m}(\theta,\phi)=P_n^{\,m}(\cos\theta)e^{im\varphi}$ are the spherical harmonics and $P_n^{\,m}(\cos\theta)$ are the associated Legendre polynomials, n and m are the integers, and ω is the circular frequency. The substitution of Eq. (16) into Eqs. (10)–(13) provides us

$$\left[\nabla_2^{\,2} + 1 - \frac{\eta^2}{\xi^2}\right]u_n = 0 \tag{17}$$

$$\left(\nabla_2^{\,2} - i\omega^{-1} - \frac{a_4^{\,2}}{\xi^2}\right)T_n - i\varepsilon\left[\left(\frac{\partial}{\partial\xi} + \frac{4\bar{\beta}-1}{2\xi}\right)w_n + \frac{\bar{\beta}(n^2+n)}{\xi}v_n\right] = 0 \tag{18a}$$

$$-\left(\nabla_2^{\,2} + 1 - \frac{a_2^{\,2}}{\xi^2}\right)v_n + \left[(1+c_3)\frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{a_1^{\,2}}{\xi^2}\right]w_n - \frac{\bar{\beta}\omega^{-1}}{\xi}T_n = 0 \tag{18b}$$

$$\left(c_4\nabla_2^{\,2} + 1 - \frac{a_3^{\,2}}{\xi^2}\right)w_n + n(n+1)\left[(1+c_3)\frac{1}{\xi}\frac{\partial}{\partial\xi} - \frac{a_1^{\,2}}{\xi^2}\right]v_n - \omega^{-1}\left(\frac{\partial}{\partial\xi} + \frac{3-4\bar{\beta}}{2\xi}\right)T_n = 0 \tag{18c}$$

where $\nabla_2^{\,2}=(1/\xi)(d/d\xi)(\xi(d/d\xi))$ and $\xi=r\omega$. Here $a_1^{\,2}$, $a_2^{\,2}$, $a_3^{\,2}$, $a_4^{\,2}$, and η^2 defined as

$$a_1^{\,2} = \frac{3+2(c_1+c_2)-c_3}{2}, \quad a_2^{\,2} = \frac{9+4(c_1(n^2+n)+c_2-c_1)}{4}$$

$$a_3^{\,2} = \frac{c_4+4(n^2+n)+8(c_1-c_3+c_2)}{4}, \quad a_4^{\,2} = \frac{1+4\bar{K}(n^2+n)}{4}$$

$$\eta^2 = \tfrac{1}{4}[9+2(n^2+n-2)(c_1-c_2)]$$

The uncoupling of equations for the displacement potential u_n from v_n, w_n, and T_n indicates the existence of two distinct classes of vibrations in the considered sphere. The solution of Eq. (17) for u_n corresponds to the toroidal modes of vibrations, which remains unaffected due to thermal variations. The frequencies of such toroidal modes of vibrations will be same as obtained by Chau [14].

Solution of Eq. (17) has the form

$$u_n(r\omega) = r^{-1/2}\sum_{n=1}^{\infty} B_{n1}J_{\eta}(r\omega), \quad n \geq 1 \tag{19}$$

where $\eta^2=\frac{1}{4}[9+2(n^2+n-2)(c_1-c_2)]>0$ and J_η is the Bessel function of first kind.

Thus the expression for potential function $\psi(r,\theta,\phi,t)$ in Eq. (16) becomes

$$\psi(r,\theta,\phi,t) = r^{-1/2}\sum_{n=1}^{\infty} B_{n1}J_{\eta}(r\omega)S_n^{\,m}(\theta,\phi)e^{i\omega t}, \quad n \geq 1 \tag{20}$$

where B_{n1} are arbitrary constants determined from the boundary conditions.

4.1 Extended Power Series Method. In the case of solid sphere, $r=0$ (i.e., $\xi=0$) is a regular singular point and all the coefficients of the differential equations (18*a*)–(18*c*) are finite, single valued, and continuous in the interval $0\leq r\leq R$, thereby satisfying all the necessary conditions to have series expressions. The Fröbenius power series method is used to solve the coupled system of partial differential equations (18*a*)–(18*c*). In order to apply the method of Fröbenius to solve Eqs. (18*a*)–(18*c*), one looks for power series of the type

$$w_n(\xi) = \sum_{k=0}^{\infty} A_k(s)(\xi)^{s+k}, \quad v_n(\xi) = \sum_{k=0}^{\infty} B_k(s)(\xi)^{s+k}$$
$$T_n(\xi) = \sum_{k=0}^{\infty} D_k(s)(\xi)^{s+k} \tag{21}$$

where s is the eigenvalue and A_k, B_k, and D_k are the unknown coefficients to be determined.

Substituting the solution (21) in Eqs. (18*a*)–(18*c*) and equating to zero the coefficients of lowest powers of ξ (here ξ^{s-2}, which corresponds to $k=0$) in the resulting system of three coupled equations, we obtain following system of equations for

$$(s^2 - a_4^{\,2})D_0 = 0$$
$$\lfloor(1+c_3)s + a_1^{\,2}\rfloor A_0 - (s^2 - a_2^{\,2})B_0 = 0 \tag{22}$$
$$(c_4 s^2 - a_3^{\,2})A_0 + n(n+1)\lfloor(1+c_3)s - a_1^{\,2}\rfloor B_0 = 0$$

A requirement for the existence of nontrivial solution of the Eq. (22) for $k=0$ results in the following system of indicial equations:

$$s^2 = \frac{1+4\bar{K}(n^2+n)}{4} \tag{23a}$$

$$s^4 - As^2 + C = 0 \tag{23b}$$

where the coefficients A and C are given by

$$A = \frac{a_3^{\,2} + c_4 a_2^{\,2} - n(n+1)(1+c_3)^2}{c_4}, \quad C = \frac{a_2^{\,2}a_3^{\,2} - n(n+1)a_1^{\,4}}{c_4}$$

The solution of indicial Eqs. (25) are obtained as

$$s_1 = \left(\frac{A+\sqrt{A^2-4C}}{2}\right)^{1/2}, \quad s_2 = \left(\frac{A-\sqrt{A^2-4C}}{2}\right)^{1/2}, \quad s_3 = a_4 \tag{24}$$

with $s_{j+3}=-s_j$, $j=1,2,3$. For the axisymmetric case the roots of Eq. (23) are distinctive real, which simplifies the analysis greatly. In case of solid sphere $r=0$ (i.e., $\xi=0$) is part of the domain of consideration and hence all the field functions (solutions) must be bounded at $\xi=0$. Therefore, we take those values of $s_j(j=1,2,3,4,5,6)$, which satisfy the radiation condition that $\mathrm{Re}(s_j)\geq 0$. Moreover, it is also assumed that any two of the three real roots do not differ by an integer.

For the choice of roots of the indicial equations, the system of Eq. (22) leads to

$$A_0(s_j) = \begin{cases} 1, & j=1,2 \\ 0, & j=3, \end{cases} \qquad B_0(s_j) = Q_B(s_j)A_0(s_j)$$
$$D_0(s_j) = \begin{cases} 0, & j=1,2 \\ 1, & j=3 \end{cases} \tag{25}$$

where

$$Q_B(s_j) = \frac{(1+c_3)s_j + a_1^{\,2}}{s_j^{\,2} - a_2^{\,2}} = -\frac{c_4 s_j^{\,2} - a_3^{\,2}}{n(n+1)[(1+c_3)s_j - a_1^{\,2}]} \quad (j=1,2) \tag{26}$$

Again equating to zero the coefficients of next lowest degree term ξ^{s-1}, which corresponds to $k=1$ and using Eqs. (24) and (25), we obtain

$$A_1(s_j) = \frac{\Delta_1(s_j)}{\Delta(s_j)}D_0(s_j), \quad B_1(s_j) = \frac{\Delta_2(s_j)}{\Delta(s_j)}D_0(s_j) \quad (j=1,2,3) \tag{27}$$

$$D_1(s_j) = \varepsilon\left(\frac{\left(s_j + \frac{(4\bar{\beta}-1)}{2}\right) + \bar{\beta}(n^2+n)Q_B(s_j)}{\left((s_j+1)^2 - \frac{1+4\bar{K}(n^2+n)}{4}\right)}\right)A_0(s_j)$$

$$(j=1,2,3)$$

where

$$\Delta_1(s_j) = \left(-\bar{\beta}p_{22}(s_j) - p_{12}(s_j)\left(s_j + \frac{3-4\bar{\beta}}{2}\right)\right)/\omega$$

$$\Delta_2(s_j) = \left(p_{11}(s_j)\left(s_j + \frac{3-4\bar{\beta}}{2}\right) - \bar{\beta}p_{21}(s_j)\right)/\omega \tag{28}$$

$$\Delta(s_j) = p_{11}p_{22} + p_{12}p_{21}$$

Here the elements $p_{ij}(i,j=1,2)$ are given by

$$p_{11} = (1+c_3)(s+1) + a_1^2, \quad p_{12}(s_j) = (s_j+1)^2 - a_2^2$$

$$p_{21}(s_j) = c_4(s_j+1)^2 - a_3^2 \tag{29}$$

$$p_{22}(s_j) = n(n+1)\{(1+c_3)(s_j+1) - a_1^2\}$$

Upon substituting the solution (21) in Eq. (18) and then equating the coefficients of $(\xi)^{s+k}$ equal to zero, we obtain the recurrence relations among the coefficients $A_k(s_j)$, $B_k(s_j)$, and $D_k(s_j)$ for $k \geq 2$ as given in Appendix.

Thus the general solution of Eq. (18) has the form

$$\{w_n(\xi),v_n(\xi),T_n(\xi)\} = \sum_{j=1}^{3}\sum_{k=o}^{\infty} C_{jk}\{A_k(s_j),B_k(s_j),D_k(s_j)\}(\xi)^{s_j+k} \tag{30}$$

where $s_j(j=1,2,3)$ are assumed to satisfy the radiation condition and $\{A_k(s_j),B_k(s_j),D_k(s_j)\}$ are eigenvectors corresponding to the eigenvalues s_j and integer k. Here C_{jk} are arbitrary constants to be evaluated by using the boundary conditions. Consequently, the potential functions w, G, and T are written from Eq. (16) by using Eq. (30) as

$$\{w,G,T\}(r,\theta,\phi,t) = r^{-1/2}\sum_{n}\sum_{j=1}^{3}\sum_{k=o}^{\infty} C_{njk}\{A_k(s_j),B_k(s_j),D_k(s_j)\} \times (r\omega)^{s_j+k}P_n^{\,m}(\cos\,\theta)e^{i(m\phi+\omega t)} \tag{31}$$

$$\psi(r,\theta,\phi,t) = r^{-1/2}\sum_{n=1}^{\infty} B_{n1}J_\eta(r\omega)P_n^{\,m}(\cos\,\theta)e^{i(\omega t+m\phi)}$$

The four unknowns B_{n1} and $C_{njk}(j=1,2,3)$ can be evaluated by four boundary conditions at the boundary of solid sphere, namely, three traction-free conditions and fourth one being thermally insulated or isothermal boundary condition.

Upon using Eqs. (25) and (27) and recurrence relations given in Appendix, the solution (31) can be written as

$$w = r^{-1/2}\sum_{n=0}^{\infty}\left[\sum_{j=1}^{2}\sum_{k=0}^{\infty} C_{nj(2k)}A_{2k}(s_j)(r\omega)^{s_j+2k} + \sum_{k=1}^{\infty} C_{n3(2k-1)}A_{2k-1}(s_3)(r\omega)^{s_3+2k-1}\right]PF^*$$

$$G = r^{-1/2}\sum_{n=1}^{\infty}\left[\sum_{j=1}^{2}\sum_{k=0}^{\infty} C_{nj(2k)}B_{2k}(s_j)(r\omega)^{s_j+2k} + \sum_{k=1}^{\infty} C_{n3(2k-1)}A_{2k-1}(s_3)(r\omega)^{s_3+2k-1}\right]PF^*$$

$$T = r^{-1/2}\sum_{n=0}^{\infty}\left[\sum_{j=1}^{2}\sum_{k=1}^{\infty} C_{nj(2k-1)}D_{2k-1}(s_j)(r\omega)^{s_j+2k-1} + \sum_{k=0}^{\infty} C_{n3(2k)}D_{2k}(s_3)(r\omega)^{s_3+2k}\right]PF^*$$

$$\psi = r^{-1/2}\sum_{n=1}^{\infty} B_{n1}J_\eta(\xi)PF^* \tag{32}$$

where

$$F^{\bullet} = \exp\{i(m\phi+\omega t)\}N, \; P = P_n(\cos\,\theta)$$

and

$$N = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}$$

Here N is known as normalization number. The displacements can be written from Eqs. (9) by using expression (32) therein. The temperature gradient and stresses are obtained as

$$T_{,r} = \left\{\sum_{n=0}^{\infty}\sum_{j=1}^{3} C_{nj0}\left(s_j+\frac{1}{2}\right)PD_0(r\omega)^{s_j-1} + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{3} C_{njk}\left(s_j+k+\frac{1}{2}\right)D_{k+1}P(r\omega)^{s_j+k}\right\}r^{-1/2}F^* \tag{33}$$

$$\sigma_{rr} = r^{-1/2}\left\{\sum_{j=1}^{3}\left(\sum_{n=0}^{\infty} C_{nj0}\left(\frac{4c_3-c_4}{2}+c_4 s_j\right)A_0(s_j) + \sum_{n=1}^{\infty} C_{nj0}n(n+1)c_3B_0(s_j)\right)P(r\omega)^{s_j-1} + \sum_{k=0}^{\infty}\sum_{j=1}^{3}\left[\left(\sum_{n=1}^{\infty} C_{njk}n(n+1)c_3B_{k+1}(s_j) + \sum_{n=0}^{\infty} C_{njk}\left(\left(\frac{4c_3-c_4}{2} + c_4(s_j+k+1)\right)A_{k+1}(s_j) - \omega^{-1}D_k(s_j)\right)\right)P\right](r\omega)^{s_j+k}\right\}F^{\bullet} \tag{34a}$$

$$\sigma_{r\theta} = -r^{-1/2}\left[\sum_{j=1}^{3}\left\{\sum_{n=0}^{\infty} C_{nj0}A_0(s_j) + \sum_{n=1}^{\infty} C_{nj0}\left(\frac{3}{2}-s_j\right)B_0(s_j)\right\} \times \bar{P}P'(r\omega)^{s_j-1} + \sum_{k=0}^{\infty}\sum_{j=1}^{3}\left\{\sum_{n=0}^{\infty} C_{njk}A_{k+1}(s_j) + \sum_{n=1}^{\infty} C_{njk}\left(\frac{1}{2}-s_j-k\right)B_{k+1}(s_j)\right\}\bar{P}P'(r\omega)^{s_j+k} + \sum_{n=1}^{\infty} imB_{n1}FP/\bar{P}\right]F^{\bullet} \tag{34b}$$

$$\sigma_{r\phi} = r^{-1/2}\Bigg[\sum_{j=1}^{3}\Bigg\{\sum_{n=0}^{\infty} C_{nj0}A_0(s_j) + \sum_{n=1}^{\infty} C_{nj0}\left(\frac{3}{2}-s_j\right)B_0(s_j)\Bigg\} \times P/\bar{P}(r\omega)^{s_j-1} + \sum_{k=0}^{\infty}\sum_{j=1}^{3}\Bigg\{\sum_{n=0}^{\infty} C_{njk}A_{k+1}(s_j) + \sum_{n=1}^{\infty} C_{njk}\left(\frac{1}{2}-s_j-k\right)B_{k+1}(s_j)\Bigg\}P/\bar{P}(r\omega)^{s_j+k} + \sum_{n=1}^{\infty} imB_{n1}F\bar{P}P'\Bigg]F^{\bullet} \qquad (34c)$$

where

$$\bar{P} = \sin\theta, \quad P' = \frac{\partial P_n(\cos\theta)}{\partial(\cos\theta)}$$
$$F = \omega\left(J_{\eta+1}(r\omega) - \left(\eta - \frac{3}{2}\right)\frac{1}{r\omega}J_\eta(r\omega)\right) \qquad (35)$$

5 Secular Dispersion Relations

We assume that for a spherically isotropic, thermally conducting solid sphere the traction-free and insulated (or isothermal) boundary conditions (Eq. (15)) hold. On applying the boundary conditions (Eq. (15)) we obtain a system of four homogeneous linear algebraic equations, which will have a nontrivial solution if and only if the determinant of the coefficients B_{n1}, C_{njk}, ($j=1,2,3$) vanishes. This requirement of nontrivial solution leads to a following determinant equations for thermally insulated sphere. It is mentioned that the integer m which appears in the spherical harmonics and represents the non-axisymmetric motion ($m\neq 0$) of the sphere is not included in the frequency equations as explained in [15] for a thin isotropic spherical shell. The equations for different cases are discussed below.

Case I. For $k=0$, $n>0$, the secular equations are obtained as

$$E_{11}E_{32} - E_{12}E_{31} = 0, \quad E_{23} = 0 \qquad (36a)$$

$$R\omega J_{\eta+1}(R\omega) - \left(\eta - \tfrac{3}{2}\right)J_\eta(R\omega) = 0 \qquad (36b)$$

$$(1-\mu^2)\frac{dP_n(\mu)}{d\mu} - mP_n(\mu) = 0 \qquad (36c)$$

where $\mu=\cos\theta$

$$E_{11} = \left[\left(n(n+1)c_3Q_B(s_1) + \left(\frac{4c_3-c_4}{2} + c_4s_1\right)\right)\right]A_0(s_1)(R\omega)^{s_1}$$
$$E_{12} = \left[\left(n(n+1)c_3Q_B(s_2) + \left(\frac{4c_3-c_4}{2} + c_4s_2\right)\right)\right]A_0(s_2)(R\omega)^{s_2}$$
$$E_{31} = \left[1 + \left(\frac{3}{2}-s_1\right)Q_B(s_1)\right]A_0(s_1)(R\omega)^{s_1} \qquad (36d)$$
$$E_{32} = \left[1 + \left(\frac{3}{2}-s_2\right)Q_B(s_2)\right]A_0(s_2)(R\omega)^{s_2}$$

The secular dispersion relation (36*a*) governs three-dimensional free vibrations of a sphere, which are independent of thermal variations. Also $E_{23}=0$ is identically satisfied for the eigenvalue s_3 defined in Eq. (24). Equation (36*b*) corresponds to first class of vibrations, which are also called toroidal modes and will be discussed in Sec. 5.2 in the following analysis.

Case II. For $k>0$, $n=0$, the secular equation for thermally insulated sphere is given by

$$\det(\bar{E}'_{ij}) = 0, \quad i,j = 1,2,3 \qquad (37a)$$

where

$$\bar{E}'_{11} = \left[\left(\left(\frac{4c_3-c_4}{2} + c_4(s_1+k+1)\right)A_{k+1}(s_1) - \omega^{-1}D_k(s_1)\right)\right] \times (R\omega)^{s_1+k}$$

$$\bar{E}'_{21} = \begin{cases} \left(s_1+k+\dfrac{1}{2}\right)D_{k+1}(s_1)(R\omega)^{s_1+k} & \text{for thermally insulated} \\ D_k(s_1)(R\omega)^{s_1+k} & \text{for isothermal} \end{cases}$$

$$\bar{E}'_{31} = A_{k+1}(s_1)(R\omega)^{s_1+k} \qquad (37b)$$

The elements $\bar{E}'_{lj}$ ($j=2,3$) of the determinant Eq. (37*a*) can be obtained by just replacing s_1 in $\bar{E}'_{lj}(l=1,2,3)$ with s_j ($j=2,3$). The secular dispersion relation (37*a*) governs three-dimensional free vibration of a sphere under stress-free isothermal/thermally insulated conditions prevailing at its surface.

Case III. For $k>0$, $n>0$, the secular equation for thermally insulated/isothermal sphere are again given by

$$\det(\bar{E}_{ij}) = 0, \quad i,j = 1,2,3 \qquad (38a)$$

where

$$\bar{E}_{11} = \left[\left(n(n+1)c_3B_{k+1}(s_1) + \left(\frac{4c_3-c_4}{2} + c_4(s_1+k+1)\right)A_{k+1}(s_1) - \omega^{-1}D_k(s_1)\right)\right](R\omega)^{s_1+k}$$

$$\bar{E}_{21} = \begin{cases} \left(s_1+k+\dfrac{1}{2}\right)D_{k+1}(s_1)(R\omega)^{s_1+k} & \text{for thermally insulated} \\ D_k(s_1)(R\omega)^{s_1+k} & \text{for isothermal} \end{cases}$$

$$\bar{E}_{31} = \left[A_{k+1}(s_1) + \left(\tfrac{1}{2}-s_1-k\right)B_{k+1}(s_1)\right](R\omega)^{s_1+k} \qquad (38b)$$

Here the elements $\bar{E}_{lj}$ ($j=2,3$) of the determinant Eq. (38*a*) can be obtained by just replacing s_1 in $\bar{E}_{lj}(l=1,2,3)$ with s_j, ($j=2,3$). The secular dispersion relation (38*a*) governs three-dimensional free vibrations of a sphere under stress-free thermally insulated/isothermal conditions prevailing at its surface.

5.1 Spheroidal Vibrations (*S*-Modes). The secular Eqs. (36*a*), (37*a*), and (38*a*) governs the spheroidal vibrations (*S*-mode) for $n>0$, $k=0$ (case I), $n=0$, $k>0$ (case II) and $n>0$, $k>0$ (case III), respectively. These relations contain complete information regarding frequency and other characteristics of the spheroidal modes of vibrations in a transradially isotropic sphere. The spheroidal vibrations for $n=0$ in transversely isotropic sphere has been studied by Buchanan and Ramirez [16].

5.2 Toroidal Vibrations (*T*-Modes). The secular dispersion Eq. (36*b*) provides us

$$\tan(R\omega) = \frac{3R\omega}{3-(R\omega)^2} \quad \text{for } n=1 \qquad (39a)$$

$$R\omega J_{\eta+1}(R\omega) - \left(\eta - \tfrac{3}{2}\right)J_\eta(R\omega) = 0 \quad \text{for } n>1 \qquad (39b)$$

Clearly these modes do not depend on thermal variations as expected. The Eqs. (39*a*) and (39*b*) agree with Love [17] and are characterized by the absence of radial component of displacement. For homogeneous isotropic solid we have

Table 1 Physical data for zinc and cobalt crystals

Quantity	Units	Zinc	Cobalt	Reference
c_{11}	N m^{-2}	1.628×10^{11}	3.071×10^{11}	
c_{12}	N m^{-2}	0.362×10^{11}	1.650×10^{11}	
c_{13}	N m^{-2}	0.508×10^{11}	1.027×10^{11}	
c_{33}	N m^{-2}	0.627×10^{11}	3.581×10^{11}	
c_{44}	N m^{-2}	0.770×10^{11}	1.510×10^{11}	
β_1	N m^{-2} deg^{-1}	5.75×10^{6}	7.04×10^{6}	Dhaliwal and Singh [18]
β_3	N m^{-2} deg^{-1}	5.17×10^{6}	6.90×10^{6}	
K_1	W m^{-1} deg^{-1}	1.24×10^{2}	0.690×10^{2}	
K_3	W m^{-1} deg^{-1}	1.24×10^{2}	0.690×10^{2}	
ε	—	0.0221	0.0129	
ω^*	s^{-1}	5.01×10^{11}	1.88×10^{12}	

$$c_{11}=c_{33}=\lambda+2\mu,\quad c_{12}=c_{13}=\lambda,\quad c_{44}=\mu,\quad \beta_1=\beta=\beta_3 \tag{40}$$
$$K_1=K=K_3$$

so that $\eta=n+(1/2)$ and the secular equation (39*b*) reduces to

$$(n-1)J_{n+1/2}(\omega R)-(\omega R)J_{n+3/2}(\omega R)=0 \tag{41}$$

This agrees with the classical result obtained by Lamb [10]. These modes were also discussed in detail by Chau [14] and hence the frequencies of such toroidal modes of vibrations will be same as obtained there in case of elastokinetics.

6 Numerical Results and Discussion

In order to illustrate the analytical development we propose to carry out some numerical calculations to compute lowest frequency of some *S*-modes in the spheres made of zinc and cobalt materials, as shown in Table 1. The numerical computations have been carried out for spheroidal modes (*S*-modes) of vibrations for $k>0$, $n>0$ with the help of MATLAB files.

Due to the presence of dissipation term in heat conduction Eq. (4), the secular equations are in general complex transcendental equations, which provide us complex values of the frequency (ω). The real part of frequency (ω) gives us lowest frequency ($\Omega=\omega_R R(\rho/c_{44})^{1/2}$) and imaginary part provide us dissipation factor ($D=\omega_I R(\rho/c_{44})^{1/2}$), where $\omega_R=\mathrm{Re}(\omega)$ and $\omega_I=I_m(\omega)$ for fixed values of n and k. The numerical computations for different values of radius ($R=0.25,0.5,0.75,1.0$) by taking sufficient number of values of the Fröbenius parameter (k) in order to obtain lowest frequency and the dissipation factor of *S*-modes have been performed. The computer simulated lowest frequencies (Ω) and dissipation factor (D) in spheres of zinc and cobalt materials were presented in Figs. 1–8.

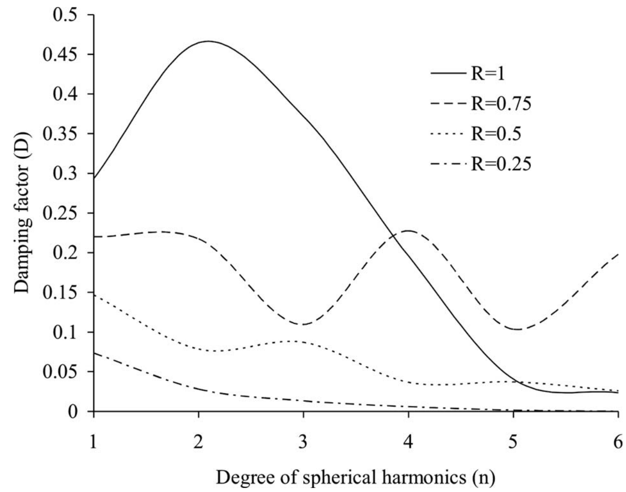

Fig. 2 Variation in dissipation versus spherical harmonics (*n*) in zinc material for different values of radius (*R*)

The variations in lowest frequency $\Omega=\omega_R R(\rho/c_{44})^{1/2}$ of a stress-free, thermally insulated sphere of zinc material were shown in Fig. 1 for various values of radius with respect to the degree of spherical harmonics (n). Figures 1 and 3 represent the variations in lowest frequency (Ω) with degree of spherical harmonics (n) for different values of radius R ($R=0.25,0.5,0.75,1.0$) for zinc and cobalt materials, respectively. For ($R=0.25,0.5,0.75$), the trends of the profiles of lowest frequency (Ω) of vibrations with degree of spherical harmonics (n) in spheres of zinc and cobalt materials are noticed to be almost similar. From Fig. 1, it is revealed that the values of lowest frequency (Ω) gradually increases up to $n=2.4$ and then decreases in range $2.4\le n\le 4$ for $R=1$ in zinc material. It is clear from Fig. 3 that the lowest frequency (Ω) increases with increasing degree of spherical harmonics (n) for $R=1$ in cobalt material.

Figures 2 and 4 represent the variations in damping factor ($D=\omega_I R(\rho/c_{44})^{1/2}$) with degree of spherical harmonics (n) for zinc and cobalt materials, respectively. For both materials, the behavior of dissipation profiles is almost similar. From both figures it is observed that the magnitude of the dissipation factor decreases

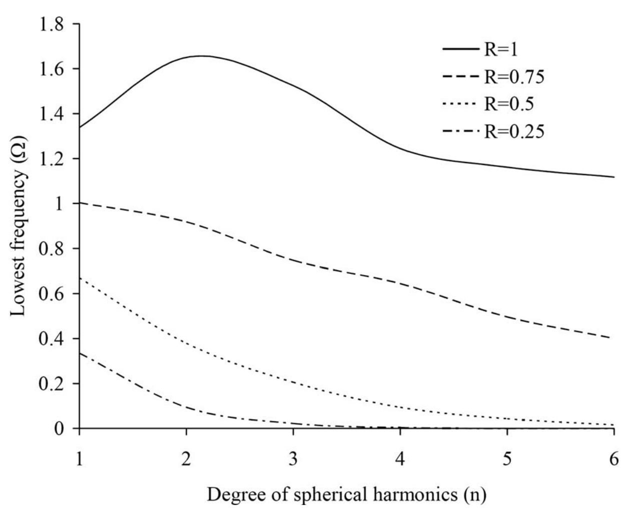

Fig. 1 Variation in lowest frequency (Ω) versus spherical harmonics (*n*) in zinc material for different values of radius (*R*)

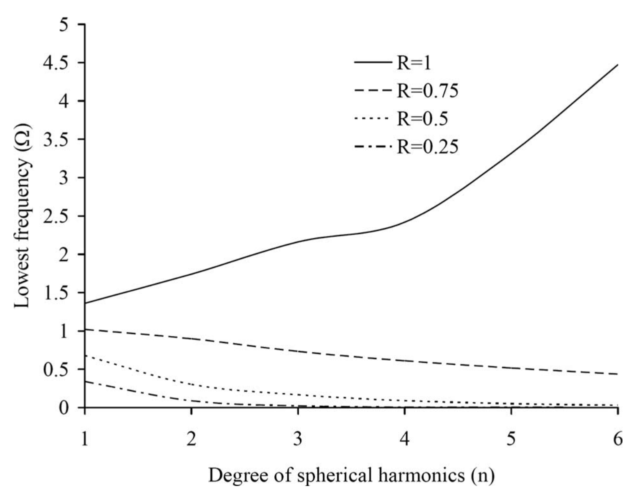

Fig. 3 Variation in lowest frequency (Ω) versus spherical harmonics (*n*) in cobalt material for different values of radius (*R*)

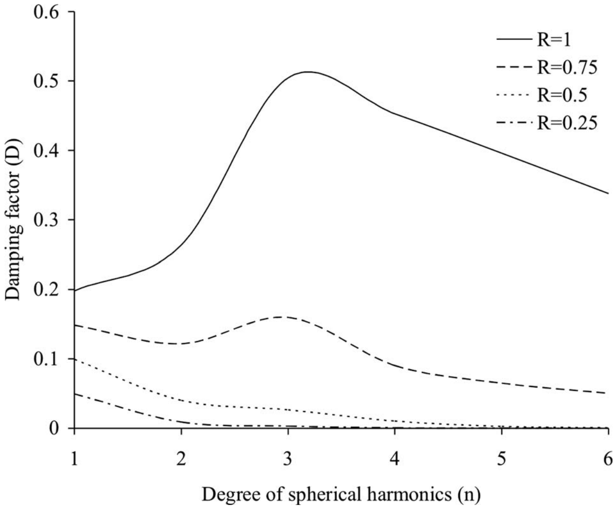

Fig. 4 Variation in dissipation (*D*) versus spherical harmonics (*n*) in cobalt material for different values of radius (*R*)

with decreasing value of radius (R) and will be negligibly small at the center of the sphere as expected. For $R=1$ the magnitude of dissipation factor increases for $1 \leq n \leq 2.4$ and decreases afterwards to show asymptotic behavior at large value of n. However, for $R=0.75$ and 0.5 the profiles of this quantity follow fluctuating trend. Almost similar behavior of profiles of this quantity is noticed for both the materials from the comparison of Figs. 2 and 4. It is evident that the behavior of the various profiles of lowest frequency and the dissipation factor of vibrations in a thermoelastic sphere made of zinc or cobalt material, become stable and steady at lower values of radius with fixed Fröbenius parameter (k) with increasing degree of spherical harmonics (n).

Figures 5 and 7 show the variations in lowest frequency (Ω) with thermal conductivity ratio $\bar{K}$ $(=K_1/K_3)$ for different degrees of spherical harmonics (n) at the surface ($R=1$) of the sphere made of zinc and cobalt materials, respectively. It is observed that the lowest frequency (Ω) increases monotonically with thermal conductivity ratio ($\bar{K}$) for $n=1$, 2, 3 in both the materials. However, the lowest frequency (Ω) decreases with increasing degree of spherical harmonics (n) and ultimately it will become independent of thermal conductivity ratio approximately after $n \geq 5$. This implies that for $n \geq 5$ all frequency intervals but not all normal modes contribute equally to the thermal conductivity in contrast to the energy content, to which all modes make equal contributions, so that the energy content is dominated by the highest frequencies or shortest wave modes. Figures 6 and 8 present the variations in dissipation factor (D) with thermal conductivity ratio ($\bar{K}$) for different values of spherical harmonics degree (n) in both zinc and

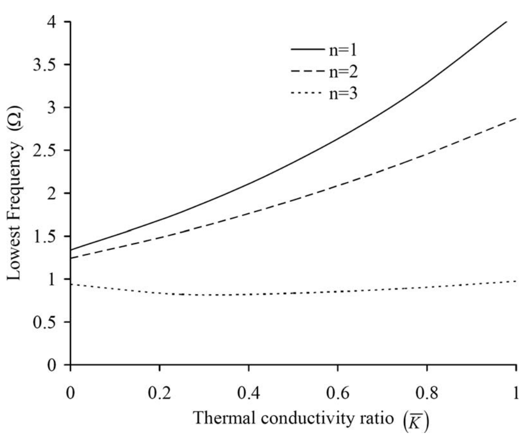

Fig. 5 Variation in lowest frequency (Ω) versus thermal conductivity ratio ($\bar{K}$) for different spherical harmonics (*n*) in zinc material for radius *R*=1

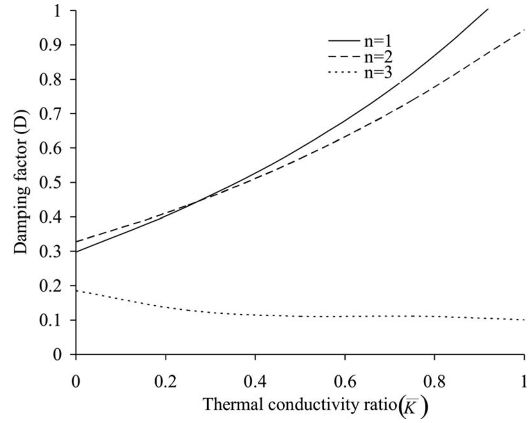

Fig. 6 Variation in dissipation (*D*) versus thermal conductivity ratio ($\bar{K}$) for different spherical harmonics (*n*) in zinc material for radius *R*=1

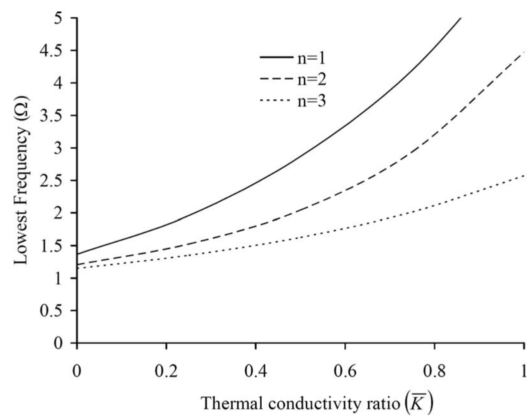

Fig. 7 Variation in lowest frequency (Ω) versus thermal conductivity ratio ($\bar{K}$) for different spherical harmonics (*n*) in cobalt material for radius *R*=1

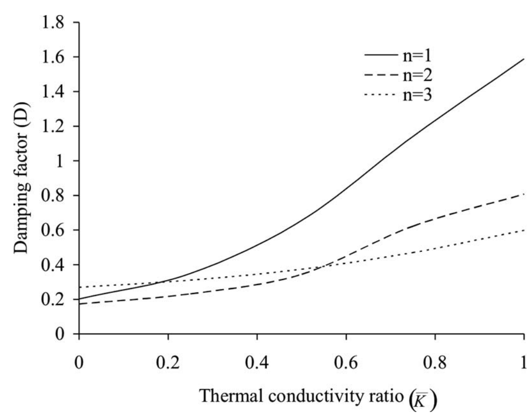

Fig. 8 Variation in dissipation (*D*) versus thermal conductivity ratio ($\bar{K}$) for different spherical harmonics (*n*) in cobalt material for radius *R*=1

cobalt materials, respectively. It is observed that the profiles of the dissipation factor (D) follow similar trend as that of lowest frequency (Ω) in Figs. 5 and 7 with respect to thermal conductivity ratio ($\bar{K}$) for $n=1$, 2, and 3 with the exception that the profiles corresponding to $n=1,2$ exhibit crossover point at $\bar{K}=0.3$ in case of zinc material and those corresponding to $n=1,3$ and $n=2,3$ show the crossovers at $\bar{K}=0.2$ and $\bar{K}=0.58$, respectively for cobalt material. Moreover, the magnitude of the dissipation factor (D) in zinc material is large for $n=2$ as compared with that of $n=1$ in $0 \leq \bar{K} \leq 0.2$, but this trend gets reversed after $\bar{K} \geq 0.2$. Similarly, the magnitude of dissipation factor (D) in cobalt material is large in $0 \leq \bar{K} \leq 0.18$ for $n=3$ as compared with that of $n=1,2$ and it lies between that of $n=1$ and $n=2$ in the range $0.18 \leq \bar{K} \leq 0.58$ before it becomes significantly small as compared with that of $n=1$ and 2 for $\bar{K} \geq 0.58$ thereafter. The close inspection of Figs. 5–8 reveals that both lowest frequency and dissipation factor of vibrations have minimum values when the sphere is nonconducting ($K_1=0$) along transradial direction at the surface of the sphere ($R=1$). The magnitudes of these quantities increase with increasing transradial thermal conductivity to attain their maximum values when the value of transradial thermal conductivity equals that of radial thermal conductivity ($K_1=K_3$) at the surface of the sphere for each degree of spherical harmonics. This clearly depicts the effect of anisotropy of material thermal conductivity on the vibrations of spheres under consideration. The increase in lowest frequency of vibrations with thermal conductivity ratio also ascertain the fact of their explicit relation in crystalline structures due to coupling effects of phonon etc. as expected. Because, if there are no phonons, all materials would be acoustic insulators and also the thermal conductivity increases with the average particle velocity since that increases the forward transport of energy.

7 Conclusions

After simplifying the system of governing equations of motion and heat conduction of three-dimensional linear coupled thermoelasticity for a transradially isotropic solid sphere with the help of Helmholtz decomposition theorem, the extended power series (Matrix Fröbenius) method is successfully used to solve exactly the resulting system of equations. It is noticed that the toroidal vibrations (T-modes) get decoupled from rest of the motion and remain independent of temperature variations as expected and the corresponding results agree with Love [17], Lamb [10], and Chau [14]. Due to thermal variations, dissipation of energy is caused, because of the damping term present in the heat conduction equation. The lowest frequency of spherical vibrations is noticed to be significantly affected due to temperature variations in both zinc and cobalt materials. The thermal conductivity exhibits significant effect on the lowest frequency and the dissipation factor profiles in zinc and cobalt materials. The lowest frequency and the dissipation factor increase in linear fashions with respect to thermal conductivity of the material along transradial to that along radial directions. This study may find applications in aerospace, navigation, geophysics tribology and other industries where spherical structures are in frequent use. This will also be a benchmark to test and check the validity of numerical methods such as finite element method, boundary element method, etc.

Acknowledgment

The authors are thankful to the University Grant Commission (UGC) New Delhi for providing financial assistance via project Grant No. F-30-236/2004(SR) to complete this work. The suggestions of reviewers for the improvement of this work are also thankfully acknowledged.

Appendix

The recurrence relations connecting the coefficients $A_k(s_j)$, $B_k(s_j)$, and $D_k(s_j)$ for $k \geq 2$ are obtained as

$$A_{k+2}(s_j) = -\lambda_0(s_j)B_{k+2}(s_j) + \lambda_1(s_j)D_{k+1}(s_j) - \lambda_2(s_j)A_k(s_j) \tag{A1}$$

$$B_{k+2}(s_j) = \lambda_0'(s_j)A_{k+2}(s_j) - \lambda_1'(s_j)D_{k+1}(s_j) - \lambda_2'(s_j)B_k(s_j) \tag{A2}$$

$$D_{k+2}(s_j) = \lambda_0''(s_j)A_{k+1}(s_j) + \lambda_1''(s_j)B_{k+1}(s_j) + \lambda_2''(s_j)D_k(s_j) \tag{A3}$$

where the coefficients $\lambda_i(s_j)$, $\lambda_i'(s_j)$, $\lambda_i''(s_j)$, $i=0,1,2$, $j=1,2,3$, and $k=0,1,2,3,\ldots$ are given by

$$\lambda_0(s_j) = n(n+1)\{(1+c_3)(s_j+k+2) - a_1{}^2\}/L_1(s_j)$$

$$\lambda_1(s_j) = \left(s_j + k + \frac{5-4\bar{\beta}}{2}\right)/\omega L_1(s_j)$$

$$\lambda_2(s_j) = 1/L_1(s_j) \tag{A4}$$

$$\lambda_0'(s_j) = (1+c_3)(s_j+k+2) + a_1{}^2/L_2(s_j)$$

$$\lambda_1'(s_j) = \bar{\beta}/\omega L_2(s_j) \tag{A5}$$

$$\lambda_2'(s_j) = 1/L_2(s_j)$$

$$\lambda_0''(s_j) = i\varepsilon\left((s_j+k+1) - \frac{1-4\bar{\beta}}{2}\right)/L_3(s_j)$$

$$\lambda_1''(s_j) = i\varepsilon\bar{\beta}(n^2+n)/L_3(s_j) \tag{A6}$$

$$\lambda_2''(s_j) = i\omega/L_3(s_j)$$

$$L_1(s_j) = c_4(s_j+k+2)^2 - a_3{}^2$$

$$L_2(s_j) = (s_j+k+2)^2 - a_2{}^2 \tag{A7}$$

$$L_3(s_j) = ((s+k+2)^2 - a_4{}^2)$$

References

[1] Maiti, M., 1975, "Stress in Anisotropic Sphere," J. Eng. Mech., **101**, pp. 101–108.
[2] Montagner, J. P., and Anderson, D. L., 1989, "Constrained Ref. Mantle Model," Phys. Earth Planet. Inter., **58**, pp. 205–227.
[3] Chen, W. Q., 1996, "Couple Free Vibrations of Spherically Isotropic Hollow Spheres," Ph.D. thesis, Zhejiang University, Hangzhou, China, in Chinese.
[4] Schafbuch, P. J., Rizzo, F. J., and Thompson, R. B., 1992, "Eigen Frequencies of Elastic Sphere With Fixed Boundary Conditions," ASME J. Appl. Mech., **59**, pp. 458–459.
[5] Hu, H. C., 1954, "On the General Theory of Elasticity for a Spherically Isotropic Medium," Acta Sci. Sin., **3**, pp. 247–260.
[6] Chen, W. T., 1966, "On Some Problems in Spherically Isotropic Elastic Materials," ASME J. Appl. Mech., **33**, pp. 539–546.
[7] Cohen, H., Shah, A. H., and Ramakrishna, C. V., 1972, "Free Vibrations of a Spherically Isotropic Hollow Sphere," Acustica, **26**, pp. 329–333.
[8] Ding, H. J., and Chen, W. Q., 1996, "Non Axisymmetric Symmetric Free Vibrations of a Spherically Isotropic Spherical Shell Embedded in an Elastic Medium," Int. J. Solids Struct., **33**, pp. 2575–2590.
[9] Ding, H. J., Chen, W. Q., and Liu, Z., 1995, "Solutions to Equations of Vibrations of Spherical and Cylindrical Shells," Appl. Math. Mech., **16**, pp. 1–15.
[10] Lamb, H., 1881, "On the Vibrations of a Elastic Sphere," Proc. London Math. Soc., **13**, pp. 189–212.
[11] Lapwood, E. R., and Usami, T., 1996, *Free Oscillations of the Earth*, Cambridge University, Cambridge, UK.
[12] Sharma, J. N., and Sharma, P. K., 2002, "Free Vibration Analysis of Homogeneous Transversely Isotropic Thermoelastic Cylindrical Panel," J. Therm. Stresses, **25**, pp. 169–182.

[13] Chen, W. Q., Cai, J. B., Ye, G. R., and Ding, H. J., 2000, "On Eigenfrequencies of an Anisotropic Sphere," ASME J. Appl. Mech., **67**, pp. 422–424.
[14] Chau, K. T., 1998, "Toroidal Vibrations of Anisotropic Spheres With Spherical Isotropy," ASME J. Appl. Mech., **65**, pp. 59–65.
[15] Silbiger, A., 1962, "Non-Axisymmetric Modes of Vibrations of Thin Spherical Shell," J. Acoust. Soc. Am., **34**, p. 862.
[16] Buchanan, G. R., and Ramirez, G. R., 2002, "A Note on the Vibration of Transversely Isotropic Solid Spheres," J. Sound Vib., **253**(3), pp. 724–732.
[17] Love, A. E. H., 1994, *A Treatise on the Mathematical Theory of Elasticity*, Dover, New York.
[18] Dhaliwal, R. S., and Singh, A., 1980, *Dynamic Coupled Thermoelasticity*, Hindustan, New Delhi, India.